Homework 4

February 15, 2017

1. Let $X(t)$ be a pure birth continuous time Markov chain. Assume that

$$P(\text{an event happens in } (t, t+h)|X(t) = odd) = \lambda_1 h + o(h)$$
$$P(\text{an event happens in } (t, t+h)|X(t) = even) = \lambda_2 h + o(h),$$

where $o(h)/h \to 0$ as $h \to 0$. Take $X(0) = 0$. Find the following probabilities: $P_1(t) = P(X(t) = odd)$ and $P_2(t) = P(X(t) = even)$.
Hint: Derive the differential equations

$$P_1'(t) = -\lambda_1 P_1(t) + \lambda_2 P_2(t),\ P_2'(t) = \lambda_1 P_1(t) - \lambda_2 P_2(t).$$

2. Under the above problem specification, derive $E[X(t)]$.

3. Consider two independent Poisson processes $X(t)$ and $Y(t)$ such that $E(X(t) = \lambda t$ and $E(Y(t) = \mu t$. Let two successive events of $X(t)$ process occur at times T and $T' > T$, so that $X(t) = X(T)$ for $T \leq t < T'$ and $X(T') = X(T)+1$. Define $N = Y(T') - Y(T)$ be the random variable that denotes the number of events of the $Y(t)$ process between T and T'. Show that $P(N(t) = m) = \frac{\lambda}{\lambda+\mu}\left(\frac{\mu}{\lambda+\mu}\right)^m, \quad m = 0, 1, 2, 3, ...$

4. Suppose a bank only has one person in the counter serving customers. People come to bank according to Poisson process at rate λ per hour, and there is only one server with the service time being exponentially distributed with rate μ. The waiting room can have as many as people as we want. Compute π_n, the steady state probability that there are exactly n customers in the system, $n = 0, 1, ...$

5. A cable car starts off with n riders. The times between successive stops of the car are independent exponential random variables with rate λ. At each stop one rider gets off. This takes no time and no additional riders get on. After a rider gets off the car, he or she walks home. Independently of all else, the walk takes an exponential time with rate μ.
(a) What is the distribution of the time at which the last rider departs the car?
(b) Suppose the last rider departs the car at time t. What is the probability that all the other riders are home by that time?

6. Let $\{N(t) : t \geq 0\}$ is a Poisson process with rate λ. Define a sequence of nonnegative, independent and identically distributed random variables $\{D_i : i \geq 1\}$ independent

from $N(t)$. A stochastic process $\{Y(t) : t \geq 0\}$ is a compound Poisson process if $Y(t) = \sum_{i=1}^{N(t)} D_i$.

(a) Find $E[Y(t)]$, $Cov(Y(t), Y(s))$.

(b) Customers arrive at an automatic teller machine (ATM) in accordance with a Poisson process with rate 12 per hour. The amount of money withdrawn on each transaction is a random variable with mean \$30 and standard deviation \$50. (A negative withdrawal means that money was deposited.) Suppose that the machine is in use 15 hours per day. Formulate this scenario as a compound Poisson process to find the expected amount of money withdrawn per day. Can you provide an approximate probability that the total daily withdraw is greater than \$5000?

7. At each (discrete) time point $n = 0, 1, 2..$, a number Y_n of particles enters a chamber, where the $Y_n, n \geq 0$, are independent and identically Poisson distributed with parameter λ, that is, for all $n \geq 0$, $Pr(Y_n = y) = \lambda^y \exp(-\lambda)/y!$, for $y \in \{0, 1, 2...\}$. Each particle in the chamber may decay during two successive time points with probability q (where $0 < q < 1$) which is constant over time and the same for all particles. Moreover, particles decay independently of each other and Y_n is independent of the number of particles that decay between time points $n - 1$ and n. Let X_n be the number of particles in the chamber at time n, where X_n includes particles present just after the entry of the Y_n particles.

   (1) Show that $X = \{X_n : n \geq 0\}$ is a time homogeneous Markov chain, and prove that its transition probabilities are given by

$$P(X_{n+1} = j | X_n = i) = \exp(-\lambda) \sum_{r=0}^{\min\{i,j\}} \frac{i!}{(i-r)!r!(j-r)!} (1-q)^r q^{i-r} \lambda^{j-r}.$$

   (2) Show that the limiting probabilities for the number of particles in the chamber are given by

$$\lim_{n\to\infty} P(X_n = j) = \exp(-\lambda q^{-1}) \frac{(\lambda q^{-1})^j}{j!}, \ \ j \in \{0, 1, 2...\}.$$