Homework 3

January 28, 2017

1. Assume $X = \{X_n : n \geq 0\}$ is a Markov chain and let $\{n_k : k \geq 0\}$ be an unbounded increasing sequence of positive integers. Define a new stochastic process $Y = \{Y_k : k \geq 0\}$ such that $Y_k = X_{n_k}$. Show that Y is a Markov chain. Is Y a time-homogeneous Markov chain without additional conditions?

2. Show that the Markov property is equivalent to each of the following conditions:
(1)For all $n, m \geq 1$, and all $s, x_0, ..., x_n \in \mathcal{S}$

$$P(X_{n+m} = s \,|\, X_1 = x_1, ..., X_n = x_n) = P(X_{n+m} = s \,|\, X_n = x_n).$$

(2) For all $0 \leq n_1 < \cdots < n_k \leq n$, all $m \geq 1$, $s, x_1, ..., x_k \in \mathcal{S}$

$$P(X_{n+m} = s \,|\, X_{n_1} = x_1, ..., X_{n_k} = x_k) = P(X_{n+m} = s \,|\, X_{n_k} = x_k).$$

3. Write the transition matrix of the following Markov chains.
(1)n black balls and n white balls are placed in two urns so that each urn contains n balls. At each stage one ball is selected at random from each urn and the two balls interchange. The state of the system is the number of white balls in the first urn.
(2)Consider two urns A and B containing a total of n balls. An experiment is performed in which a ball is selected at random at time t $(t = 1, ...)$ from among the totality of n balls. Then an urn is selected at random (prob. of selecting A is p) and the ball previously drawn is placed in this urn. The state of the system at each trial is the number of balls in A.

4. Consider the Markov chain with state space $\mathcal{S} = \{1, 2, 3, 4\}$ and transition matrix

$$\begin{pmatrix} 1/3 & 2/3 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Classify the states of the chain, and calculate f_{34}, the probability of absorption in state 4, starting from state 3.

5. A transition matrix $\boldsymbol{P} = ((P_{ij}))_{i,j=1}^{K}$ with finite state space $\mathcal{S} = \{1, ..., K\}$ is known to be a doubly stochastic matrix if $\sum_{i=1}^{K} P_{ij} = 1$, for all $j = 1, ..., K$. Prove that the

stationary distribution of a doubly stochastic matrix is a discrete uniform distribution.

6. Sociologists often assume that the social classes of successive generations in a family can be regarded as a Markov chain. Thus, the occupation of a son is assumed to depend only on his father's occupation and not on his grandfather's. Suppose that such a model is appropriate and that the transition probability matrix is given by

	Lower	*Middle*	*Upper*
Lower	0.40	0.50	0.10
Middle	0.05	0.70	0.25
Upper	0.05	0.50	0.45

where columns correspond to son's class and rows correspond to father's class. For such a model, what fraction of people are middle class in the long run?

7. Consider the following random walk with state space $\mathcal{S} = \{0, 1, 2, 3, 4\}$ and transition matrix:

$$P_{i,i+1} = p, P_{i,i-1} = 1 - p = q, \text{ for } i = 1, 2, 3;\ P_{0,0} = P_{4,4} = 1.$$

Find $d(k) = \mathrm{E}[\text{time to absorption into states 0 or 4—initial state is } k]$. Prove that

$$\begin{aligned} d(k) &= \frac{k}{q-p} - \frac{r}{q-p}\frac{(1-(q/p)^r)}{(1-(q/p)^r)}, \text{ if } p \neq 1/2 \\ &= k(4-k) \text{ if } p = 1/2. \end{aligned}$$

8. Consider an insurance company that earns \$1 per day (from interest) with probability p, but on each day, independent of the past, might suffer a claim against it for the amount \$2 with probability $q = 1 - p$. Whenever such a claim is suffered, \$2 is removed from the reserve of money. Thus on the n-th day, the net income for that day is exactly Δ_n as in the gamblers ruin problem: 1 with probability p, -1 with probability q. Find P(the fortune of the company hits 10\$ before becoming -10).

9. Consider a Markov chain with three states $\mathcal{S} = \{1, 2, 3\}$ and transition matrix

$$\begin{pmatrix} 1-2p & 2p & 0 \\ p & 1-p & p \\ 0 & 2p & 1-2p \end{pmatrix}$$

where $0 < p < 0.5$. Classify the states of the chain. Calculate $p_{ii}^{(n)}$, $i = 1, 2, 3$. Calculate the mean recurrence time μ_i, $i = 1, 2, 3$, of the states.