Homework 1

January 12, 2017

1. Consider real-valued random variables $A_i, B_i, i = 1, ..., k$, such that $E(A_i) = E(B_i) = 0$ and $Var(A_i) = Var(B_i) = \sigma_i^2 > 0$ for $i = 1, ..., k$. Moreover, assume they are mutually uncorrelated, that is, $E(A_i A_l) = E(B_i B_l) = 0$, for $i \neq l$, and $E(A_i B_l) = 0$, for all i, l. Define the stochastic process $X = \{X_t : t \in \mathcal{R}\}$ by $X_t = \sum_{i=1}^{k}(A_i \cos(\omega_i t) + B_i \sin(\omega_i t))$, where ω_i are real constants $i = 1, ..., k$. Show that X is weakly stationary.

2. Consider a discrete-time real-valued stochastic process $X = \{X_n : n \geq 1\}$ defined by $X_n = \cos(nU)$, where U is uniformly distributed on $(-\pi, \pi)$. Show that X is weakly stationary but not strongly stationary.

3. Consider a weakly stationary process $X = \{X_t : t \in \mathcal{R}\}$ with zero mean and unit variance. Find the correlation function of X if the spectral density function f of X is given by:
   (a) $f(u) = 0.5 \exp(-|u|), u \in \mathcal{R}$
   (b) $f(u) = \phi(\alpha^2 + u^2)^{-1}$, $u \in \mathcal{R}$
   (c) $f(u) = \frac{1}{2}\sigma(\pi\alpha)^{-1} \exp(-u^2/(4\alpha))$, $u \in \mathcal{R}$.

4. Show that strong and weak stationarity are equivalent for a Gaussian process.

5. By definition, a continuous-time real-valued stochastic process $X = \{X_t : t \in \mathcal{R}\}$ is called a Markov process if for all n, for all $x, x_1, ..., x_n$, and all increasing sequences $t_1 < \cdots < t_n$ of index points,

   $$Pr(X_{t_n} \leq x \,|\, X_{t_1} = x_1, ..., X_{t_{n-1}} = x_{n-1}) = Pr(X_{t_n} \leq x \,|\, X_{t_{n-1}} = x_{n-1})$$

   Let Z be a real valued Gaussian process. Show that Z is a Markov process if and only if $E(Z_{t_n} \leq x \,|\, Z_{t_1} = x_1, ..., Z_{t_{n-1}} = x_{n-1}) = E(Z_{t_n} \leq x \,|\, Z_{t_{n-1}} = x_{n-1})$.

6. Show that any stochastic process $X = \{X_n : n = 0, ...\}$ with independent increments is a Markov process.

7. Let $W = \{W_t : t \geq 0\}$ is a Brownian motion (see classnotes for the definition). Show that a Brownian motion can be viewed as a Gaussian process with mean 0 and $Cov(W_s, W_t) = \min\{s, t\}$.

8. Show that for a Brownian motion $E(|W_s - W_t|^{2n}) = C_n|s - t|^n$, $C_n = \frac{2n!}{n!n!}$.